International Journal of Theoretical Physics, Vol. 15, No. 12 (1976), pp. 949–959

A Theory of Generally Invariant Lagrangians for the Metric Fields. II

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Received: 5 *November* 1975

Abstract

The second-order generally invariant Lagrangians for the metric fields on any n-dimensiona manifold are studied as certain special coordinate functions on a space of jets. The number of independent Lagrangians of this type is determined. The dimensions $n = 3$ and $n = 4$ are examined in detail with the help of a computer.

1. *Introduction*

The problem of finding all invariant functions composed of a geometric object field and its first two derivatives has become important since the birth of general relativity theory. Using an algebraic approach, some authors were able to obtain relatively complete results for the metric tensor fields on four-dimensional base manifolds (Géhéniau and Debever, 1956; Géhéniau, 1956; Petrov, 1966), for the scalar–tensor and vector–tensor field theories (Horndeski and Lovelock, 1972; Lovelock, 1974), etc. Related questions, as for instance the existence and uniqueness of certain invariant tensor fields, were also discussed.

On the other hand, a new general method of description of all generally invariant functions depending on the components of a geometric object and their derivatives up to an arbitrary order, has recently been proposed in terms of the theory of fiber bundles (Krupka and Trautman, 1974; Krupka, 1974). The method has been applied to the problem of second-order invariants of the metric fields in the first part of this work (Krupka, to appear). The purpose of this paper is to complete the previous results by showing the number of independent invariant functions, and by their explicit description in the cases of three- and four-dimensional base manifolds.

In my preceding paper, I have characterized the domain $T_n^2(R^{n*} \odot R^{n*})$

and the invariance group $L_n{}^3$ of the second-order generally invariant functions. We have described the natural action of $L_n{}^3$ on $T_n{}^2(R^{n*} \odot R^{n*})$, and found a collection of vector fields spanning the corresponding Lie algebra of vector fields. In certain local coordinates g^{ik}, $\Gamma_{i,jk}$, R_{ijkl}, $S_{i,jkl}$ around any regular point of $T_n{}^2(R^{n*} \odot R^{n*})$ (i.e., the point satisfying $\det(g^{ij}) \neq 0$) the vector fields may be chosen in the form

$$\begin{aligned}
\overset{+}{\Xi}_{ij} &= \frac{\partial}{\partial g^{ij}} - (g_{ip}R_{jqrs} + g_{jp}R_{iqrs}) \frac{\partial}{\partial R_{pqrs}}, && i \leqslant j \\
\overset{-}{\Xi}_{ij} &= (g_{ip}R_{jqrs} - g_{jp}R_{iqrs}) \frac{\partial}{\partial R_{pqrs}}, && i < j \\
\Xi^{i,jk} &= \frac{\partial}{\partial \Gamma_{i,jk}}, && j \leqslant k \\
\Xi^{i,jkl} &= \frac{\partial}{\partial S_{i,jkl}}, && j \leqslant k \leqslant l
\end{aligned} \tag{1.1}$$

In these formulas, n denotes the dimension of the base manifold, $i, j, p, q, \ldots = 1, 2, \ldots, n$, and the summation convention is used. We shall deal with the vector-field system (1.1) referred to as the fundamental vector-field system on the manifold $T_n{}^2(R^{n*} \odot R^{n*})$, and with its integral functions called the second-order generally invariant functions of the metric fields, or just the second-order invariants.

2. *The Rank of the Fundamental Vector-Field System*

In the context of the theory of vector-field systems, well elaborated for our purposes in a book by Hermann (Hermann, 1968), the rank of the fundamental vector-field system at a regular point $(g^{ij}, \Gamma_{i,jk}\, R_{ijkl}, S_{i,jkl})$ of the manifold $T_n{}^2(R^{n*} \odot R^{n*})$ is defined as the rank of the matrix formed by the coefficients in (1.1) standing at the base vector fields

$$\frac{\partial}{\partial g^{ij}}, \quad \frac{\partial}{\partial \Gamma_{i,jk}}, \quad \frac{\partial}{\partial R_{ijkl}}, \quad \frac{\partial}{\partial S_{i,jkl}}$$

and evaluated at the point. It is most important to know the rank of (1.1) at the maximal points, i.e., at the points of $T_n{}^2(R^{n*} \odot R^{n*})$ where the rank is maximal.

The form of the vector fields (1.1) shows that the rank (always assumed to be taken at a maximal point) is given as

$$r = r' + \tfrac{1}{2}n(n+1) + n \cdot \binom{n+1}{2} + n \cdot \binom{n+2}{3} \tag{2.1}$$

where r' is the rank of the matrix

$$
\begin{array}{cc}
 & \begin{array}{cc} \dfrac{\partial}{\partial R_{ijij}},\ i<j\rightarrow & \dfrac{\partial}{\partial R_{ijkl}},\ (ij)\neq(kl)\rightarrow \end{array} \\
\begin{array}{c} \Xi_{ij}^{-} \\ i<j \\ \downarrow \end{array} & \left(\begin{array}{c:c} \quad \Delta_n \quad & \qquad\qquad\qquad \end{array}\right)
\end{array}
\tag{2.2}
$$

and the other terms denote the number of independent vector fields $\partial/\partial g^{ij}$, $\partial/\partial\Gamma_{i,jk}$, $\partial/\partial S_{i,jkl}$. In what follows we work with the square matrix Δ_n of dimension $(1/2)n\cdot(n-1)$ formed by the coefficients in Ξ_{ij}^{-} standing at the vector fields $\partial/\partial R_{ijij}$ (no summation). The remaining block matrix in (2.2) has not been written down explicitly.

Let us establish the rank of the fundamental vector-field system (1.1) for one-, two-, and three-dimensional base manifolds, and then prove a proposition concerning the general n-dimensional case.

For $n = 1$ the fundamental vector-field system has the rank equal to the dimension of the manifold $T_1{}^2(R^{1*}\odot R^{1*})$, which is equal to 3. Hence there is no nontrivial second-order invariant.

The case $n = 2$ has been discussed in the first part of this work. We have seen that $\Xi_{12}^{-} = 0$ and that at each regular point the rank of the fundamental vector-field system is equal to the number of the coordinates g^{ij}, $\Gamma_{i,jk}$, $S_{i,jkl}$, that is to 17. The dimension of the manifold $T_n{}^2(R^{n*}\odot R^{n*})$ is equal to 18, and we have exactly one nontrivial independent second-order invariant.

Let us examine the case $n = 3$. To show that at some points of the manifold $T_3{}^2(R^{3*}\odot R^{3*})$ the rank of the fundamental vector-field system (1.1) is equal to the number of vector fields in (1.1), it suffices to verify that the determinant det Δ_3 of the matrix Δ_3 (2.2) does not vanish identically. For $n = 3$ there are six independent coordinates R_{ijkl}, say $R_{1212}, R_{1313}, R_{2323}, R_{1213}, R_{1223}, R_{1323}$. The coefficients at the vector fields $\partial/\partial R_{1212}$, $\partial/\partial R_{1313}$, $\partial/\partial R_{2323}$ form the matrix Δ_3:

$$
\left(\begin{array}{cc}
0 & g_{12}R_{1313} - g_{11}R_{1323} - g_{13}R_{1213} \\
g_{13}R_{1212} + g_{11}R_{1223} - g_{12}R_{1213} & 0 \\
g_{12}R_{1223} + g_{13}R_{1212} - g_{22}R_{1213} & g_{13}R_{1323} + g_{33}R_{1213} - g_{23}R_{1313} \\
\multicolumn{2}{c}{g_{22}R_{1323} - g_{12}R_{2323} - g_{23}R_{1223}} \\
\multicolumn{2}{c}{g_{23}R_{1323} - g_{33}R_{1223} - g_{13}R_{2323}} \\
\multicolumn{2}{c}{0}
\end{array}\right)
$$

One can easily check that the coefficient in the polynomial det Δ_3 standing at $R_{1212}R_{1313}R_{1223}$ is equal to $g_{13}(g_{23}^2 - g_{12}g_{33})$ and therefore does not vanish identically. We conclude that there exist some values of the coordinates g^{ij}, R_{ijkl} where det $\Delta_3 \neq 0$. At the points with these coordinates, the rank (2.1) of the fundamental vector-field system (1.1) is maximal, and is equal to 57. Since the manifold $T_3{}^2(R^{3*}\odot R^{3*})$ is 60-dimensional, we have exactly three nontrivial independent second-order invariants of the metric fields.

Let us now study the general case $n \geqslant 3$. We wish to show that the rank of the fundamental vector-field system is equal to the number of vector fields in (1.1). It suffices to prove that there exists a point of the manifold $T_n{}^2(R^{n*}\odot R^{n*})$ at which $\det \Delta_n \neq 0$. The matrix elements of Δ_n (2.2) are given by

$$\Delta_{ij,\,pq} = g_{jp}R_{iqpq} - g_{ip}R_{jqpq} - g_{jq}R_{ippq} + g_{iq}R_{jppq} \tag{2.3}$$

where $1 \leqslant i < j \leqslant n$, $1 \leqslant p < q \leqslant n$. Note that the matrix Δ_n has the form

$$\Delta_n = \left(\begin{array}{c|c} \Delta_{n-1} & \\ \hline & \Lambda_n \end{array}\right)$$

where

$$\Delta_2 = 0$$

$$\Lambda_n = \begin{pmatrix} \Delta_{1n,\,1n} & \Delta_{1n,\,2n} & \cdots & \Delta_{1n,\,n-1n} \\ \Delta_{2n,\,1n} & \Delta_{2n,\,2n} & \cdots & \Delta_{2n,\,n-1n} \\ \cdot & \cdot & \cdots & \cdot \\ \Delta_{n-1n,\,1n} & \Delta_{n-1n,\,2n} & \cdots & \Delta_{n-1n,\,n-1n} \end{pmatrix}$$

and

$$\Delta_{ij,\,ij} = 0$$

$$\Delta_{in,\,jn} = g_{jn}R_{injn} - g_{in}R_{jnjn} - g_{nn}R_{ijjn}$$

($1 \leqslant i, j \leqslant n-1$). From (2.3) it follows that all matrix elements of Δ_n containing g_{nn} lie in the submatrix Λ_n. Let us write down the coefficient in the determinant $\det \Delta_n$ standing at $(g_{nn})^{n-1}$. It is given as

$$\det \Delta_{n-1} \cdot \det \begin{pmatrix} 0 & R_{122n} & R_{133n} & \cdots & R_{1n-1n-1n} \\ R_{211n} & 0 & R_{233n} & \cdots & R_{2n-1n-1n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ R_{n-111n} & R_{n-122n} & R_{n-133n} & \cdots & 0 \end{pmatrix} \tag{2.4}$$

Evidently, all elements R_{ijkl} appearing in the second determinant (2.4) are independent coordinates on the manifold $T_n{}^2(R^{n*}\odot R^{n*})$: The only possible dependencies could arise from the well-known identities

$$R_{ijkl} + R_{iljk} + R_{iklj} = 0$$

which are, however, automatically satisfied by all coordinates R_{ijkl} having two equal indices. We deduce that the second determinant (2.4) does not vanish identically. If now the rank of the fundamental vector-field system were not equal, at a point, to the number of vector fields in (1.1), the determinant $\det \Delta_n$ would identically vanish and the coefficient (2.4) at $(g_{nn})^{n-1}$ would be equal to zero. As the factor $\det \Delta_{n-1}$ does not contain, by definition, the coordinates R_{ijjn}, our assumption would give $\det \Delta_{n-1} = 0$ identically, a contradiction with the three-dimensional case. The rank of the fundamental vector-field system (1.1) must therefore be equal to the number

$$r = n \cdot \left[n + \binom{n+1}{2} + \binom{n+2}{3} \right] \tag{2.5}$$

The dimension of the manifold $T_n{}^2(R^{n*} \odot R^{n*})$ being equal to

$$\dim T_n{}^2(R^{n*} \odot R^{n*}) = \tfrac{1}{2}n(n+1) + \tfrac{1}{2}n^2(n+1) + \tfrac{1}{4}n^2(n+1)^2 \tag{2.6}$$

we obtain the maximal number of nontrivial independent second-order invariants subtracting (2.5) from (2.6).

Our discussion can be summarized as follows.

For $n \geqslant 3$ the rank of the fundamental vector-field system (1.1) at its maximal points is given by (2.5). There exist exactly

$$M_n = \tfrac{1}{12} n(n-1)(n-2)(n+3) \tag{2.7}$$

nontrivial independent second-order invariants of the metric fields. For $n = 1$ the rank is equal to 3, and there exists no nontrivial second-order invariant. For $n = 2$ the rank is equal to 17, and there is just one nontrivial independent second-order invariant.

Thus for $n = 3$ ($n = 4$, $n = 5$) each basis of the second-order invariants of the metric fields consists of $M_3 = 3$ ($M_4 = 14$, $M_5 = 40$) members.

Not to complicate the language, we shall speak of the signature of a point $j_0{}^2 f \in T_n{}^2(R^{n*} \odot R^{n*})$, having in mind the signature of the matrix g^{ij} of its coordinates (see Krupka, to appear). We note that the results of this section are independent of the signature.

3. *Generally Invariant Functions*

Having obtained the number of the generally invariant functions of any flat local coordinate system on the manifold $T_n{}^2(R^{n*} \odot R^{n*})$, we may start to determine a basis of them.

The form of the fundamental vector-field system (1.1) implies that each generally invariant function defined on an open domain of regular points in $T_n{}^2(R^{n*} \odot R^{n*})$ depends on the coordinates g^{ij} and R_{ijkl} only. Further rough information about these functions can be obtained from the action of the group $L_n{}^3$ on $T_n{}^2(R^{n*} \odot R^{n*})$ written in the coordinates g^{ij}, R_{ijkl} (Krupka, to appear). The group action suggests a construction, similarly as in various classical considerations, of certain generally invariant functions in the form of homogeneous polynomials; we shall call these invariants the canonical invariants. The simplest examples are the canonical invariants

$$\begin{aligned} I &= g^{ik}g^{jl}R_{ijkl} \\ J &= g^{ip}g^{jq}g^{kr}g^{ls}R_{ijkl}R_{pqrs} \\ K &= g^{ac}g^{bi}g^{dp}g^{jl}g^{kr}g^{qs}R_{abcd}R_{ijkl}R_{pqrs} \end{aligned} \tag{3.1}$$

well known in geometry and relativity.

A question arises as to whether all generally invariant functions belonging to a flat local coordinate system on $T_n{}^2(R^{n*} \odot R^{n*})$ can be taken as the canonical invariants. If we consider a domain of regular points in $T_n{}^2(R^{n*} \odot R^{n*})$ around a maximal point of the fundamental vector-field system (1.1) as an analytic manifold then the Frobenius complete integrability theorem ensures the existence of a flat local coordinate system on the domain such that all its generally invariant functions are analytic in g^{ij} and R_{ijkl}. Associated with these generally invariant functions are uniquely determined homogeneous polynomials which have to be generally invariant by themselves. It follows that there exists a basis of generally invariant functions consisting of canonical invariants.

The problem of a description of all generally invariant functions is thus reduced to the problem of finding their basis formed by the canonical invariants. Further simplification follows from the fact that some points of $T_n{}^2(R^{n*} \odot R^{n*})$ having the same signature can be joined by a transformation from $L_n{}^3$. Hence all our considerations to determine such a basis can be made on a neighborhood of a selected point, say, with the coordinates g^{ij} in a canonical form (i.e., g^{ij} equal to 0 if $i \neq j$, and g^{ii} equal to 1 or -1). After having found a basis one can transfer it by the group transformation onto other open sets of points with the same signature.

Let us assume that we have an open set U in $T_n{}^2(R^{n*} \odot R^{n*})$ of maximal and regular points with a given signature. Let U be covered by a flat local coordinate system, let I_ι, $\iota = 1, 2, \ldots, M_n$ (2.7) denote the corresponding basis of generally invariant functions. Then the 1-forms

$$dI_\iota = \frac{\partial I_\iota}{\partial g^{ij}}\, dg^{ij} + \frac{\partial I_\iota}{\partial R_{ijkl}}\, dR_{ijkl} = \frac{\partial I_\iota}{\partial R_{pqrs}}\, (dR_{pqrs} + 2g_{ip}R_{jqrs}dg^{ij})$$

are linearly independent on U, and the rank of the matrix $(\partial I_\iota / R_{pqrs})$ is equal to M_n at every point of U. If U contains a point whose matrix g^{ij} is in a canonical form then it is not too difficult to see that any change of the signature leaves, at suitable points of $T_n{}^2(R^{n*} \odot R^{n*})$, the rank M_n unchanged. The results to be established can now be formulated more precisely.

Around any maximal point of the manifold $T_n{}^2(R^{n*} \odot R^{n*})$ one can always choose a flat local coordinate system so that all its generally invariant functions are the canonical invariants. If a system I_ι, $\iota = 1, 2, \ldots, M_n$, of canonical invariants forms a basis of generally invariant functions on an open domain of maximal points with a given signature, then it forms a basis of generally invariant functions on any other open domain of maximal points, independently of their signature.

Let us return to the problem of finding a basis for the second-order generally invariant functions of the metric fields. We know the maximal number of functionally independent second-order invariants and have guaranteed that they can be taken as the canonical invariants. The canonical invariants may even be looked for at the points $(g^{ij}, \Gamma_{i,jk}, R_{ijkl}, S_{i,jkl})$ satisfying $g^{ij} = \delta^{ij}$, where δ^{ij} stands for the Kronecker symbol. This suggests an effective procedure of constructing a basis and the corresponding flat local coordinate system. Firstly, we choose some canonical invariants I_ι, $\iota = 1, 2, \ldots, M_n$, and consider them at a point where $g^{ij} = \delta^{ij}$. Secondly, on differentiating we obtain the matrix $(\partial I_\iota/\partial R_{ijkl})$, substitute the numerical values for the variables R_{ijkl}, and form the corresponding numerical matrix $(\partial I_\iota/\partial R_{ijkl})_0$. Thirdly, we compute the rank of $(\partial I_\iota/\partial R_{ijkl})_0$. If it is equal to M_n then the canonical invariants I_ι define the desired basis and we are done. Otherwise we have to choose another point (i.e., other numerical values for the variables R_{ijkl}) or start with another collection of canonical invariants. In this way the theory of generally invariant functions is reduced to computations of determinants. We remark that the order of determinants we are to consider in practice is, as a rule, a large number: For $n = 4$ it is less than or equal to $M_4 = 14$. It is therefore necessary to calculate the determinants with the aid of computers.

4. *Example:* $n = 3$

Applying the results of Sections 2 and 3 to three-dimensional manifolds, we obtain the following:

Around each maximal point of the manifold $T_3^2(R^{3*} \odot R^{3*})$ the canonical invariants (3.1) form a basis of generally invariant functions.

To verify this assertion, it suffices to consider the invariants (3.1) at a point where $g^{ij} = \delta^{ij}$, and show that the matrix

$$\begin{pmatrix} \dfrac{\partial I}{\partial R_{1212}} & \dfrac{\partial I}{\partial R_{1313}} & \dfrac{\partial I}{\partial R_{2323}} & \dfrac{\partial I}{\partial R_{1213}} & \dfrac{\partial I}{\partial R_{1223}} & \dfrac{\partial I}{\partial R_{1323}} \\ \dfrac{\partial J}{\partial R_{1212}} & \dfrac{\partial J}{\partial R_{1313}} & \dfrac{\partial J}{\partial R_{2323}} & \dfrac{\partial J}{\partial R_{1213}} & \dfrac{\partial J}{\partial R_{1223}} & \dfrac{\partial J}{\partial R_{1323}} \\ \dfrac{\partial K}{\partial R_{1212}} & \dfrac{\partial K}{\partial R_{1313}} & \dfrac{\partial K}{\partial R_{2323}} & \dfrac{\partial K}{\partial R_{1213}} & \dfrac{\partial K}{\partial R_{1223}} & \dfrac{\partial K}{\partial R_{1323}} \end{pmatrix}$$

has the rank, at a point, equal to 3. Writing

$$I = \sum_{i,j} R_{ijij}, \qquad J = \sum_{i,j,k,l} R_{ijkl}^2, \qquad K = \sum_{i,j,k,l,p,q} R_{ijkj} R_{ilql} R_{kpqp}$$

we easily obtain for the determinant of the submatrix consisting of the first three columns the expression

$$\det\begin{pmatrix} 1 & 0 & 0 \\ R_{1212} & R_{1313}-R_{1212} & R_{2323}-R_{1212} \\ R_{1lml}R_{1pmp}+R_{2lml}R_{2pmp} & R_{3lml}R_{3pmp}-R_{2lml}R_{2pmp} & R_{3lml}R_{3pmp}-R_{1lml}R_{1pmp} \end{pmatrix}$$

where we sum over pairs of indices. This determinant does not vanish identically. On comparison with (2.7) we obtain our assertion.

In view of the result, it can be concluded that at the points where $g^{ij}=\delta^{ij}$ and the above determinant is nonzero, the functions $g^{ij}, I, J, K, R_{1213}, R_{1223}, R_{1323}$ form a flat local coordinate system for the vector-field system (1.1).

5. *Example:* $n = 4$

Let us study in detail the case of four-dimensional base manifolds. Using the same notation as before we set

$$R_{ij}=g^{kl}R_{ikjl},\qquad R^{ij}=g^{ik}g^{jl}R_{kl},\qquad R_j{}^i=g^{ik}R_{jk}$$
$$R^i_{jkl}=g^{im}R_{mjkl},\qquad R^{ij}_{kl}=g^{ip}g^{jq}R_{pqkl},\qquad R^{ijkl}=g^{ip}g^{jq}g^{kr}g^{ls}R_{pqrs}$$

and introduce the following polynomials:

$$\begin{aligned}
I_1 &= R_i{}^i\\
I_2 &= R^{ij}R_{ij}\\
I_3 &= R^{ijkl}R_{ijkl}\\
I_4 &= R^{ij}R_i{}^kR_{jk}\\
I_5 &= R^{ik}R^{jl}R_{ijkl}\\
I_6 &= R^{mjkl}R_m{}^iR_{ijkl}\\
I_7 &= R^{pqij}R^{kl}_{pq}R_{ijkl}\\
I_8 &= R^{ik}R^{jl}R_{ij}R_{kl}\\
I_9 &= R^{ip}R^{qjkl}R_{pq}R_{ijkl}\\
I_{10} &= R^{pq}R^{jk}R^l_{pqi}R^i_{jkl}\\
I_{11} &= R^{jp}R^{klrs}R^i_{prs}R_{ijkl}
\end{aligned}\tag{5.1}$$

$$I_{12} = R^{pq} R^{i}_{pmq} R^{mjkl} R_{ijkl}$$
$$I_{13} = R^{pqij} R^{klrs} R_{pqrs} R_{ijkl}$$
$$I_{14} = R^{pjkl} R^{iqrs} R_{pqrs} R_{ijkl}$$

All these polynomials, homogeneous in both g^{ij} and R_{ijkl}, satisfy the system of partial differential equations defined by the vector fields (1.1). We shall sketch the proof of the following theorem:

Theorem. Around each maximal point of the manifold $T_4{}^2(R^{4*} \odot R^{4*})$ the canonical invariants (5.1) form a basis of generally invariant functions.

To prove this assertion, we must examine, as indicated at the end of Section 3, the corresponding Jacobi matrix $(\partial I_\iota / \partial R_{ijkl})$. Its elements can easily be computed by means of a simple algebraic lemma:

Lemma. Let C^{ijkl} be any collection of numbers, where $i, j, k, l = 1, 2, \ldots, n$. There exist two uniquely determined collections of numbers, C_0^{ijkl} and C_1^{ijkl}, such that

$$C^{ijkl} = C_0^{ijkl} + C_1^{ijkl}$$
$$C_0^{ijkl} = -C_0^{jikl} = -C_0^{ijlk} = C_0^{klij}$$
$$C_0^{ijkl} + C_0^{iljk} + C_0^{iklj} = 0$$

and

$$C_1^{ijkl} R_{ijkl} = 0$$

for all values of the variables R_{ijkl}. If C^{ijkl} has the property

$$C^{ijkl} = -C^{jikl} = -C^{ijlk} = C^{klij}$$

then C_0^{ijkl} is defined by

$$C_0^{ijkl} = \tfrac{1}{3}(2C^{ijkl} - C^{iljk} - C^{iklj})$$

We shall apply the lemma to our homogeneous polynomials (5.1). Note that each of the polynomials is of the form

$$f = C^{ijkl} R_{ijkl}$$

where C^{ijkl} are polynomials in g^{ij} and R_{ijkl}. Since f is homogeneous we have

$$\frac{\partial f}{\partial R_{ijkl}} R_{ijkl} = K \cdot f = K \cdot C^{ijkl} R_{ijkl} = K \cdot C_0^{ijkl} R_{ijkl}$$

where K denotes the degree of f (in the variables R_{ijkl}). Thus the derivatives $\partial f / \partial R_{ijkl}$, where all the variables R_{ijkl} are considered as independent, are equal to the suitably symmetrized C^{ijkl}, i.e., to C_0^{ijkl}. Not all the variables

R_{ijkl} are, however, independent; we choose the following 20 independent variables among the R_{ijkl}'s:

$$\begin{gathered} R_{1212}, R_{1313}, R_{1414}, R_{2323}, R_{2424}, R_{3434}, R_{1213} \\ R_{1214}, R_{1314}, R_{2123}, R_{2124}, R_{2324}, R_{3132}, R_{3134} \\ R_{3234}, R_{4142}, R_{4143}, R_{4243}, R_{1234}, R_{1324} \end{gathered} \tag{5.2}$$

and consider all other components R_{ijkl} as functions of (5.2). Then the derivative $\partial f/\partial R_{ijkl}$ with respect to (5.2) becomes equal, up to a constant nonzero factor depending on the sequence of the indices i, j, k, l, to the function C_0^{ijkl}. Denote by $C_{0(\iota)}^{ijkl}$ the function defined in this way by the polynomial I_ι; for example

$$C_{0(8)}^{ijkl} = \tfrac{1}{4} R_{pq}(R^{pi}R^{qk}g^{jl} - R^{pj}R^{qk}g^{il} - R^{pi}R^{ql}g^{jk} + R^{pj}R^{ql}g^{ik})$$

The rank of the matrix $(\partial I_\iota/\partial R_{ijkl})$, where $\iota = 1, 2, \ldots, 14$ and R_{ijkl} runs over the set (5.2), must therefore be the same as the rank of the matrix $(C_{0(\iota)}^{ijkl})$, at each point. One can easily construct the latter matrix. According to Section 3 we consider the matrix at the point

$$\begin{gathered} R_{1212} = R_{1213} = R_{1314} = R_{2123} = R_{2324} = R_{4143} = R_{1324} = 1 \\ R_{1414} = R_{2323} = R_{2424} = R_{1214} = R_{3132} = R_{3134} \\ = R_{3234} = R_{4142} = R_{4243} = R_{1234} = 0 \\ R_{1343} = R_{3434} = R_{2124} = -1 \\ g^{ij} = \delta^{ij} \end{gathered} \tag{5.3}$$

To determine its rank we consider the squared submatrix defined by the variables $R_{2424}, R_{1213}, R_{1214}, R_{2123}, R_{2124}, R_{2324}, R_{3132}, R_{3134}, R_{3234}, R_{4142}, R_{4143}, R_{4243}, R_{1234}, R_{1324}$. The calculation by means of a computer shows that the rank is equal to the number of the polynomials (5.1), i.e., to 14, which proves our theorem.

Another result of our calculation is concerned with the particular point (5.3) of the manifold $T_4^2(R^{4*} \odot R^{4*})$:

On a neighborhood of the point (5.3), the functions $g^{ij}, I_\iota(5.1), R_{1212}, R_{1313}, R_{1414}, R_{2323}, R_{3434}, R_{2123}$ form a flat local coordinate system for the vector-field system (1.1).

Acknowledgment

The author would like to thank Dr. F. Pecka from the Department of Computing Science, Brno University, for having done all numerical calculations of Section 5 on a computer.

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